UDC 539.3

INDENTATION OF A STAMP IN A HALF-PLANE WITH CIRCULAR INCLUSIONS

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Iu. A. AMENZADE
(Baku)

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The problem of the indentation of a stamp in a half-plane with circular holes in which circular inclusions from another material are imbedded with given tightness, is solved.

1. A method of solving problems of the indentation of a stamp with a flat base into a half-plane with inclusions is given in [1]. In order to realize the method of [1] let us present the solution of the problem of a stamp in a half-plane with circular holes (the domain S_0) in which circular inclusions of radii r_k $(k = 1, 2, \ldots, m)$ from another material (the domain S_k) are imbedded with given tightnesses δr_k (Fig. 1).

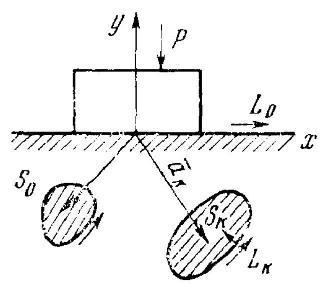

Fig. 1

In [1] the contact problem, which does not take account of friction under the stamp when the line beyond the stamp is free of forces and total contact of the stamp with the half-plane boundary occurs, is reduced to a Riemann problem with index -1, whose general solution is regular in a z-plane slit along the line L $((0, \infty), (-\infty, 0))$ and takes on the zero value of the following function at infinity:

$$F(z) = \frac{X_0(z)}{2\pi i}\int\limits_L \frac{f(t)\,dt}{X_0^+(t)(t-z)} - \frac{iP}{2\pi}X_0(z) \tag{1.1}$$

Here

$$X_0(z) = (z^2 - a^2)^{-1/2}, \qquad f(t) = 2\,[f_2^-(t) + \bar{f}_2^-(t)] + f_1^-(t) + \bar{f}_1^-(t) \tag{1.2}$$

$$f_2(z) = I_1'(z), \qquad f_1(z) = zI_1''(z) + I_2'(z) \tag{1.3}$$

$$I_1(z) = \sum_{k=1}^{m}\frac{1}{2\pi i}\int\limits_{L_k}\frac{\omega_k(t)}{t-z}\,dt, \qquad I_2(z) = -\sum_{k=1}^{m}\frac{1}{2\pi i}\int\limits_{L_k}\frac{\overline{\omega_k(t)} + \bar{t}\omega_k'(t)}{t-z}\,dt \tag{1.4}$$

where $2a$ is the stamp width, and P is the force with which the stamp is indented in the half-plane.

2. To solve the problem, let us take an auxiliary unknown function $\omega_k(t)$ in the form

$$\omega_k(t) = \sum_{\nu=-\infty}^{\infty}\alpha_{\nu k}\left(\frac{t-\bar{a}_k}{r_k}\right)^{\nu} \tag{2.1}$$

Here $a_k = d_k - ih_k$ is the affix of the center of the circle L_k and $\alpha_{0k}, \alpha_{\nu k}, \alpha_{-\nu k}$ are unknown constant coefficients which are generally complex. Substituting the function $\omega_k(t)$ into (1.4) we obtain

$$I_1'(z)\,|_{z\in S_0}=I_{10}'(z),\quad I_{10}'(z)=\sum_{k=1}^{m}\sum_{\nu=1}^{\infty}\frac{\nu-1}{r_k}\alpha_{-\nu+1,\,k}\left(\frac{r_k}{z-\bar{a}_k}\right)^{\nu} \tag{2.2}$$

$$I_1'(z)\,|_{z\in S_q}=I_{10}'(z)+\sum_{\substack{\nu=0\\(k\neq q)}}^{\infty}\frac{\nu+1}{r_q}\alpha_{\nu+1,\,q}\left(\frac{z-\bar{a}_q}{r_q}\right)^{\nu}$$

$$I_2'(z)\,|_{z\in S_0}=I_{20}'(z),\quad I_{20}'(z)=-\sum_{k=1}^{m}\sum_{\nu=1}^{\infty}\frac{\nu-1}{r_k}\Big[\bar{\alpha}_{\nu-1,\,k}- \tag{2.3}$$

$$\frac{a_k}{r_k}(\nu-2)\,\alpha_{-\nu+2,\,k}-(\nu-3)\,\alpha_{-\nu+3,\,k}\Big]\left(\frac{r_k}{z-\bar{a}_k}\right)^{\nu}$$

$$I_2'(z)\,|_{z\in S_q}=I_{20}'(z)-\sum_{\substack{\nu=1\\(k\neq q)}}^{\infty}\frac{\nu}{r_q}\Big[\bar{\alpha}_{-\nu,\,q}+\frac{a_q}{r_q}(\nu+1)\,\alpha_{\nu+1,\,q}+$$

$$(\nu+2)\,\alpha_{\nu+2,\,q}\Big]\left(\frac{z-\bar{a}_q}{r_q}\right)^{\nu-1}$$

On the basis of (1, 3), (2, 2), (2, 3), we determine on L_k from (1, 2)

$$f(t)=\sum_{k=1}^{m}\sum_{\nu=1}^{\infty}\left\{\frac{\nu-1}{r_k}(2\alpha_{-\nu+1,\,k}+l_{\nu k})\left(\frac{r_k}{t-\bar{a}_k}\right)^{\nu}+\right. \tag{2.4}$$

$$\left.\frac{\nu-1}{r_k}(2\bar{\alpha}_{-\nu+1,\,k}+\bar{l}_{\nu k})\left(\frac{r_k}{t-a_k}\right)^{\nu}\right.$$

where

$$l_{\nu k}=-\left\{\nu\alpha_{-\nu+1,\,k}-2(\nu-2)\,i\frac{h_k}{r_k}\alpha_{-\nu+2,\,k}+\bar{\alpha}_{\nu-1,\,k}-(\nu-3)\,\alpha_{-\nu+3,\,k}\right\}$$

Let us substitute (2, 4) into (1, 1) and let us take into account that

$$\sqrt{\bar{a}_k^{\,2}-a^2}=-\sqrt{a_k^{\,2}-a^2}$$

Then by virtue of the residue theorem, we find from (1, 1) after a number of manipulations

$$F(z)=\frac{1}{2}\sum_{k=1}^{m}\sum_{q=1}^{\infty}\left\{\left[\bar{P}_q(k)\left(\frac{r_k}{z-\bar{a}_k}\right)^{q}+P_q(k)\left(\frac{r_k}{z-a_k}\right)^{q}\right]+\right. \tag{2.5}$$

$$\left.\frac{1}{\sqrt{z^2-a^2}}\left[\bar{Q}_q(k)\left(\frac{r_k}{z-\bar{a}_k}\right)^{q}-Q_q(k)\left(\frac{r_k}{z-a_k}\right)^{q}\right]\right\}-\frac{iP}{2\pi\sqrt{z^2-a^2}}$$

$$\bar{P}_q(k)=\frac{q-1}{r_k}(2\alpha_{-q+1,\,k}+l_{qk})$$

$$\bar{Q}_j=(-1)^q\left(\frac{\bar{a}_k+a}{r_k}\right)^{q}\frac{\sqrt{\bar{a}_k^{\,2}-a^2}}{r_k}\sum_{\nu=q}^{\infty}\frac{(-1)^{\nu-1}(\nu-1)}{(\nu-1)!}\times$$

$$(2\alpha_{-\nu+1,\,k}+l_{\nu k})\left(\frac{r_k}{\bar{a}_k+a}\right)^{\nu}\bar{e}_{q-1}(\nu,\,k)$$

$$\bar{e}_s(\nu,\,k)=2^{s-\nu+1}\frac{(\nu-1)!}{(\nu-1-s)!}\sum_{j=0}^{\nu-1-s}C_{\nu-1-s}^{j}(2j-3)!!\,(2\nu-2s-2j-5)!!\times$$

$$\left(\frac{\bar{a}_k+a}{\bar{a}_k-a}\right)^{j}\quad((-1)!!=1,\quad(-3)!!=-1)$$

Taking account of the relationships (2.3), (2.2) and (1.3) as well as (2.5) and taking the expansions in L_q

$$\left(\frac{r_k}{t-a_k}\right)^p = \left(\frac{-r_k}{a_k-\bar{a}_q}\right)^p \sum_{\nu=0}^{\infty} (-1)^\nu C_{-p}^{\nu} \left(\frac{r_q}{a_k-\bar{a}_q}\right)^\nu \left(\frac{t-\bar{a}_q}{r_q}\right)^\nu$$

$$\left(\frac{t-a_k}{r_k}\right)^p = (-1)^p \left(\frac{a_k-\bar{a}_q}{r_k}\right)^p \sum_{\nu=0}^{p} (-1)^\nu C_p^{\nu} \left(\frac{r_q}{a_k-\bar{a}_q}\right)^\nu \left(\frac{t-\bar{a}_q}{r_q}\right)^\nu$$

$$\frac{1}{\sqrt{t^2-a^2}} = \frac{1}{\sqrt{\bar{a}_k^2-a^2}} \sum_{n=0}^{\infty} \left(\frac{r_q}{\bar{a}_q+a}\right)^n \sum_{m=0}^{n} C_{-1/2}^{m} C_{-1/2}^{n-m} \left(\frac{\bar{a}_q+a}{\bar{a}_q-a}\right)^m \left(\frac{t-\bar{a}_q}{r_q}\right)^n$$

we obtain $2m$ infinite systems of linear equations for the Fourier coefficients α_{0q}, $\alpha_{\nu q}$, $\alpha_{-\nu q}$ which we do not present because of their awkwardness.

Let us present the results of a numerical computation for $a/h = 1/3$, $r/h = 1/2$, $\bar{a}_1 = -ih$, $\varkappa_0 = \varkappa_1 = 2$, $\mu_0/\mu \ll 1$ ($\varkappa_0$, μ_0 and $\varkappa_k$, μ_k are the respective elastic constants of the materials filling the domains S_0 and S_k) and $\delta r_1 = 0$. The first three complex equations are hence retained out of the infinite systems of equations. The pressure under the stamp has been calculated at the points $x/a = 0, 0.25, 0.5, 0.75$ and 0.9 and pressure diagrams have been constructed (Fig. 2). The pressure diagram when there are no inclusions is shown for comparison by a fine continuous line.

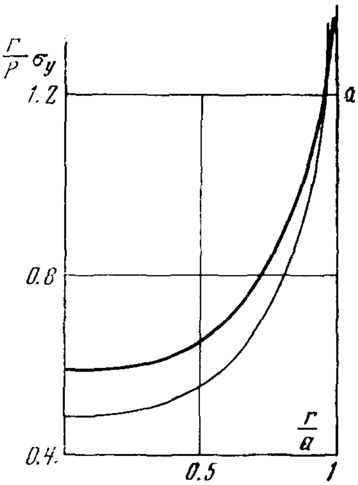

Fig. 2

REFERENCE

1. Amenzade, Iu. A., Pressure of a stamp on a half-plane with inclusions. PMM Vol. 36, № 5, 1972.

Translated by M.D.F.